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SEMI-MARKOV PROCESSES AND α-INVARIANT DISTRIBUTIONS

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A semi-Markov process is easily made Markov by adding some auxiliary random variables. This paper discusses the I-type quasi-stationary distributions of such "extended" processes, and the α-invariant distributions for the corresponding Markov transition probabilities; and we show that there is an intimate relation between the two. The results have relevance in the study of the time to "absorption" or "death" of semi-Markov processes. The particular case of a terminating renewal process is studied as an example.

semi-Markov processes	Markov renewal processes
α-recurrence	α-invariant distribution
epoch of absorption	terminating renewal process

1. Introduction

We start by explaining the historical development which has led to this paper: Kingman [8] showed that an irreducible α-positive recurrent Markov process $\{X(t)\}$ on a countable state space I and with transition probabilities $P_{ij}(t)$ possesses a left α-invariant vector $\pi = (\pi_i)_{i\in I}$, satisfying $\sum_{i\in I}\pi_i P_{ij}(t) = e^{-\alpha t}\pi_j$ for all $j \in I$, $t \geq 0$. If $\{X(t)\}$ is recurrent, $\alpha = 0$ and this adds nothing to the standard results concerning the invariant (stationary) distribution of the process. In the transient case, it is possible that $\alpha > 0$, and then the α-invariant vector has a "quasi-stationary" interpretation (cf. Vere-Jones [12]). In particular, if $\{X(t)\}$ is strictly substochastic, i.e. $\sum_{j\in I}P_{ij}(t) < 1$ for some $i \in I$ and some $t > 0$, then an extra absorbing state ∂ can be adjoined to I to make $\{X(t)\}$ stochastic. In the latter case, if p_i is the probability of ultimately reaching ∂ from i and if $\{X(t)\}$ is again irreducible α-positive recurrent, then Vere-Jones [12] and Tweedie [11] showed that there exists a unique I-type quasi-stationary (stationary conditional) distribution $\mu = (\mu_i)_{i\in I}$, satisfying $\mu_i = (\sum_{j\in I}\pi_j p_j)^{-1}\pi_i p_i$. So if $p_i = 1$ for all $i \in I$ (which is necessarily the case when I is finite) then μ is the α-invariant vector π normed to a probability.

These results do not have naive generalizations for semi-Markov processes: Cheong [2] proved that unique I-type quasi-stationary distributions exist for α-positive recurrent semi-Markov processes but he also indicated that α-invariant distributions may not exist for the semi-Markov transition functions themselves. In this paper we find semi-Markov generalizations of Kingman's and Vere-Jones' results. We shall do so by considering, not only the variable $X(t)$ which is the state of the process at time t, but also the three "auxiliary variables" $V^-(t)$ (the time since the last jump), $V^+(t)$ (the time to the next jump) and $X^+(t)$ (the state visited at the next jump) (see Çinlar [3], pp. 161-162).

We first find (Theorem 4.1) a quasi-stationary limit for the multivariate process $\{(X(t), X^+(t), V^-(t), V^+(t))\}$; as easy corollaries of this, we find quasi-stationary limits for the more familiar "backward process" $\{(X(t), V^-(t))\}$ and "forward process" $\{(X(t), X^+(t), V^+(t))\}$. Using these limiting results we can then derive α-invariant measures for the transition probabilities of both the backward and the forward process.

To illustrate our results, in the last section we consider a terminating renewal process in a slightly more general form than usual.

2. Preliminaries

Our notation closely follows Nummelin [9]. We denote $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}_+ = [0,\infty)$, $\mathcal{R} = \{B \subset \mathbf{R};\ B \text{ Borel set}\}$, $\mathcal{R}_+ = \{B \in \mathcal{R};\ B \subset \mathbf{R}_+\}$. The Lebesgue measure on $\mathcal{R}_+$ is denoted by ℓ and we write $\mathrm{d}\ell(x) = \ell(\mathrm{d}x) = \mathrm{d}x$. The indicator function of a set B is 1_B and we write $B + x = x + B = \{x + y;\ y \in B\}$. Any measure or function on $(\mathbf{R}_+, \mathcal{R}_+)$ is automatically extended to $(\mathbf{R}, \mathcal{R})$ by making it zero on the negative real axis. Let F and G be any two measures on $(\mathbf{R}_+, \mathcal{R}_+)$ and let $f: \mathbf{R}_+ \to \mathbf{R}_+$ be measurable. We write $F[a,b]$ (resp. $F(a,b), F[a,b)$) instead of $F([a,b])(F((a,b)), F([a,b)))$, and for $\lambda \in \mathbf{R}$, $x \in \mathbf{R}_+$, $B \in \mathcal{R}_+$

$$F^\lambda(B) = \int_B \mathrm{e}^{\lambda t}\mathrm{d}F(t),$$

$$f^\lambda(x) = \mathrm{e}^{\lambda x} f(x), \qquad F_B(x) = F(x + B).$$

The convolution of F and G is the measure

$$(F * G)(B) = \int_{\{x+y \in B\}} \mathrm{d}F(x)\mathrm{d}G(y)$$

and the convolution of F and f is the function

$$(F * f)(x) = \int_{\mathbf{R}_+} \mathrm{d}F(t) f(x - t).$$

We have

$$(F*G)^\lambda = F^\lambda * G^\lambda \quad \text{and} \quad (F*f)^\lambda = F^\lambda * f^\lambda.$$

Let I be a numerable set, ∂ a point not in I and $\bar{I} = I \cup \{\partial\}$. We shall be concerned with a number of stochastic processes associated with a given *semi-Markov kernel* $\bar{Q} = \{Q_{ij}\}_{i,j\in\bar{I}}$. We shall assume

$$Q_{\partial\partial}(\mathbf{R}_+) = 1.$$

Let $\bar{R} = (R_{ij})_{i,j\in\bar{I}} = \sum_{n=0}^{\infty} \bar{Q}^{*n}$ be the *Markov renewal kernel* corresponding to $\bar{Q}$. Let Q be the semi-Markov kernel $Q = (Q_{ij})_{i,j\in I}$. Then it is easy to see that the Markov renewal kernel $R = \sum_{n=0}^{\infty} Q^{*n}$ is equal to the restriction of $\bar{R}$ to $I \times I$, i.e., $R = (R_{ij})_{i,j\in I}$, and that for all $k \in I$ $R_{\partial k} = 0$. Denote $\bar{B}_i(t) = \sum_{j\in\bar{I}} Q_{ij}[0,t]$ and $(Q_{ij})_B(x) = Q_{ij}(x+B)$.

We shall assume once and for all that Q is *non-degenerate*, i.e., $Q_{ij}(0,\infty) > 0$ for some $i,j \in I$, and *irreducible*, i.e., the matrix $Q(\mathbf{R}_+)$ is irreducible.

Recall the following facts from Cheong [1] or from [9]: The *convergence parameter*

$$\alpha = \sup\{\lambda \in \mathbf{R};\, R_{ij}^{\lambda}(\mathbf{R}_+) < \infty\}$$

is independent of $i,j \in I$. The semi-Markov kernel Q is called *α-recurrent* if $R_{ij}^{\alpha}(\mathbf{R}_+) = \infty$ for some (and then all) $i,j \in I$. An α-recurrent kernel Q is said to be *α-positive recurrent* if the "mean" of the recurrence time distribution, $\int_{\mathbf{R}_+} t e^{\alpha t}\, dF_{ii}(t)$, is finite for some (and then all) $i \in I$; otherwise Q is *α-null recurrent*.

The following results in [9] will be used later (for direct Riemann integrability consult either [4] or [9]):

Proposition 2.1. *Suppose that Q is α-recurrent. Then there exist (up to a scalar multiplication) unique left and right invariant vectors, denoted respectively by $\pi = (\pi_i)_{i\in I}$ and $h = (h_j)_{j\in I}$, for the matrix $Q^\alpha(\mathbf{R}_+) = \int_{\mathbf{R}_+} e^{\alpha x} Q(dx)$. Q is α-positive recurrent iff*

$$\pi\left[\int_{\mathbf{R}_+} x e^{\alpha x} Q(dx)\right] h < \infty.$$

Proposition 2.2. *Suppose that Q is aperiodic and α-recurrent, and that the function $f: I \times \mathbf{R}_+ \to \mathbf{R}_+$ is such that $e^{\alpha(\cdot)} f$ is directly Riemann integrable w.r.t. π. Then for all $i \in I$ (we understand $f = (f_i(\cdot))_{i\in I}$ as a column vector and define $R * f : (i,t) \to \sum_{j\in I} R_{ij} * f_j(t)$),*

$$\lim_{t\to\infty} e^{\alpha t}(R*f)_i(t) = \frac{h_i\left[\pi \int_{\mathbf{R}_+} e^{\alpha x} f(x)\, dx\right]}{\pi\left[\int_{\mathbf{R}_+} x e^{\alpha x} Q(dx)\right] h}.$$

In particular, if the function $g:\mathbf{R}_+\to\mathbf{R}_+$ *is such that* $e^{\alpha(\cdot)}g$ *is directly Riemann integrable in the sense of Feller* [5, p. 361], *then for all* $i,j\in I$,

$$\lim_{t\to\infty} e^{\alpha t}R_{ij}*g(t)=\frac{h_i\pi_j\int_{\mathbf{R}_+}e^{\alpha x}g(x)\mathrm{d}x}{\pi\left[\int_{\mathbf{R}_+}x e^{\alpha x}Q(\mathrm{d}x)\right]h}.$$

3. The backward and forward processes

We shall now define the backward and forward processes mentioned in the introduction. Let $(X,T)=\{(X_n,T_n);\ n=0,1,\dots\}$ be the Markov renewal process (MRP) (for the precise definition of an MRP see [4] or [9]) corresponding to the semi-Markov kernel $\bar{Q}=(Q_{ij})_{i,j\in\bar{I}}$. For $t\in\mathbf{R}_+$ let (cf. [3])

$$M(t)=\sup\{n\in\mathbf{N};\ T_n\leq t\},$$

$$X(t)=X_{M(t)}\qquad\qquad\text{(by convention } X_\infty=\partial\text{)},$$

$$V^-(t)=t-T_{M(t)},\qquad V^+(t)=T_{M(t)+1}-t$$

and

$$X^+(t)=X_{M(t)+1}.$$

Notice that $V^-(t)\in\mathbf{R}_+$ and $V^+(t)\in\mathbf{R}_+\setminus\{0\}$.

$\{X(t);t\in\mathbf{R}_+\}$ is the semi-Markov process (SMP) corresponding to the MRP (X,T). $\{(X(t),V^-(t));\ t\in\mathbf{R}_+\}$ is called the *backward process*. The backward process is seen to be Markov with state space $\bar{I}\times\mathbf{R}_+$ and with transition probability $\bar{P}_t^{(b)}$, where

$$\bar{P}_t^{(b)}((i,x),\{j\}\times B)=$$
$$=[1-\bar{B}_i(x)]^{-1}\Big[\sum_{k\in\bar{I}}\int_{[0,t)}1_B(y)\int_{(x,t+x-y]}\mathrm{d}Q_{ik}(u)\mathrm{d}R_{kj}(t+x-y-u)(1-\bar{B}_j(y))$$
$$+\delta_{ij}1_B(t+x)(1-\bar{B}_i(t+x))\Big],\qquad i,j\in\bar{I},\quad x,t\in\mathbf{R}_+,\quad B\in\mathcal{R}_+.$$

(For our purpose it is enough to define $\bar{P}_t^{(b)}((i,x),\{j\}\times B)$ for those values of x for which $\bar{B}_i(x)<1$.) The verification of this expression for the transition probability is as follows. Let $\mathbf{P}^{(b)}_{(i,x)}$ $(i\in\bar{I},x\in\mathbf{R}_+)$ be that canonical probability on the sample space of the backward process (the infinite product space of $\bar{I}\times\mathbf{R}_+$) which corresponds to $\bar{P}_t^{(b)}$ and the start $X(0)=i$, $V^-(0)=x$. Abbreviate $\mathbf{P}^{(b)}_{(i,0)}=\mathbf{P}_i$. Then

$$\bar{P}_t^{(b)}((i,x),\{j\}\times B)=\mathbf{P}^{(b)}_{(i,x)}\{X(t)=j,V^-(t)\in B\}$$
$$=\mathbf{P}_i\{X(t+x)=j,V^-(t+x)\in B\mid V^+(0)>x\}$$

and this is equal to the right-hand side expression.

The stochastic process $\{(X(t),V^+(t))\}$ is *not* Markov, whereas $\{(X^+(t),V^-(t))\}$,

$\{(X(t), X^+(t), V^+(t))\}$ and $\{(X(t), X^+(t), V^-(t), V^+(t))\}$ are. Of these processes we will consider here in detail the *forward process* $\{(X(t), X^+(t), V^+(t))\}$. The state space of the forward process is $\bar{I} \times \bar{I} \times (0, \infty)$ and its transition probability is

$$\bar{P}_t^{(f)}((i, j, x), \{k\} \times \{l\} \times B) =$$
$$= R_{jk} * (Q_{kl})_B(t - x) + \delta_{ik}\delta_{jl} 1_B(x - t), \qquad i, j, k, l \in \bar{I}, \quad x, t \in \mathbf{R}_+, \quad B \in \mathcal{R}_+.$$

We denote by $\mathbb{P}^{(f)}_{(i,j,x)}$ that canonical probability on the sample space of the forward process (the infinite product space of $\bar{I} \times \bar{I} \times (0, \infty)$) which corresponds to $\bar{P}_t^{(f)}$ and the start $X(0) = i$, $X^+(0) = j$, $V^+(0) = x$.

Before the main theorems we prove the following lemma.

Lemma 3.1. *Suppose that Q is α-recurrent. Then for all $B \in \mathcal{R}_+$,*

$$\int_{(0,\infty)} \pi Q^\alpha [x, \infty) R^\alpha (B - x) \mathrm{d}x = \ell(B)\pi.$$

Proof. It suffices to prove the lemma for B of the form $B = [0, t]$, $(t > 0)$. We have

$$\int_{(0,\infty)} \pi Q^\alpha [x, \infty) R^\alpha ([0, t] - x) \mathrm{d}x =$$
$$= \int_{(0,t]} \pi Q^\alpha [x, \infty) \int_{[0,t-x]} \mathrm{d}R^\alpha (y) \mathrm{d}x$$
$$= \int_{[0,t)} \mathrm{d}R^\alpha (y) \int_{(0,t-y]} \pi Q^\alpha [x, \infty) \mathrm{d}x$$
$$= \int_{[0,t)} \mathrm{d}R^\alpha (y) \int_{(y,t]} \pi Q^\alpha [x - y, \infty) \mathrm{d}x$$
$$= \int_{(0,t]} \mathrm{d}x \int_{[0,x)} \pi Q^\alpha [x - y, \infty) \mathrm{d}R^\alpha (y)$$
$$= \int_{(0,t]} \mathrm{d}x \int_{[0,x)} \pi [Q^\alpha (\mathbf{R}_+) - Q^\alpha [0, x - y)] \mathrm{d}R^\alpha (y)$$
$$= \int_{(0,t]} \mathrm{d}x \pi [R^\alpha [0, x) - Q^\alpha * R^\alpha [0, x)]$$
$$= \int_{(0,t]} \mathrm{d}x \pi \Delta [0, x) = t\pi;$$

here Δ is the matrix valued measure satisfying $\Delta_{ij}(B) = \delta_{ij} 1_B(0)$ for all $i, j \in I$ and all $B \in \mathcal{R}$.

Henceforth the following *basic assumption* is made: Q is *non-degenerate, irreducible, aperiodic* and *α-positive recurrent.*

4. Quasi-stationary distributions

In this section we give our main results. First we derive the I-type quasi-stationary distributions for the backward and forward processes. Then we find α-invariant distributions for the corresponding transition probabilities.

We assumed that

$$Q_{\partial\partial}(\mathbf{R}_+) = 1,$$

i.e. ∂ is an *absorbing state* for the MRP (X, T). The *epoch of absorption* of the SMP $\{X(t)\}$ is defined by

$$\tau = \inf\{t \in \mathbf{R}_+ : X(t) = \partial\} \quad \text{(by convention } \inf\emptyset = \infty).$$

We denote the probability of *ultimate absorption* from state $i \in \bar{I}$ by

$$p_i = \mathbf{P}_i\{\tau < \infty\},$$

and assume that $p_i > 0$ for some $i \in I$.

The vector $\bar{p} = (p_i)_{i \in \bar{I}}$ satisfies (see Lemma 4.1 of [9])

$$p_i = \sum_{j \in \bar{I}} Q_{ij}(\mathbf{R}_+) p_j, \quad i \in I,$$

$$p_\partial = 1.$$

By using irreducibility it is easy to see that either $p_i = 1$ for all $i \in I$ or $p_i < 1$ for all $i \in I$. Note that our assumption $p_i > 0$ for some $i \in I$ implies $p_i > 0$ for all $i \in I$.

Theorem 4.1. *The quasi-stationary limit*

$$\mu_{ij}(A \times B) = \lim_{t \to \infty} \mathbf{P}_k\{X(t) = i, X^+(t) = j, V^-(t) \in A, V^+(t) \in B \mid t < \tau < \infty\}$$

$(i \in I, j \in \bar{I}, A, B \in \mathcal{R}_+, B \subset (0, \infty))$ exists independently of $k \in I$ provided that the function $x \to Q_{ij}(x + B) 1_A(x)$ is Riemann integrable, and the function

$$(m, x) \to \mathrm{e}^{\alpha x} \sum_{l \in \bar{I}} Q_{ml}(x, \infty) p_l = \mathrm{e}^{\alpha x} f_m(x)$$

is directly Riemann integrable w.r.t. π. We have

$$\mu_{ij}(A \times B) = \left[\pi \int_{\mathbf{R}_+} \mathrm{e}^{\alpha x} f(x) \mathrm{d}x\right]^{-1} \pi_i p_j \int_A \mathrm{e}^{\alpha x} Q_{ij}(x + B) \mathrm{d}x.$$

Proof.

$$\mathbf{P}_k\{X(t) = i, X^+(t) = j, V^-(t) \in A, V^+(t) \in B, \tau < \infty\} =$$

$$= \int_{[0,t] \cap A} \mathrm{d}R_{ki}(t - x) Q_{ij}(x + B) p_j = R_{ki} * ((Q_{ij})_B 1_A)(t) p_j.$$

By Proposition 2.2,

$$\lim_{t\to\infty} e^{\alpha t} \mathbf{P}_k \{X(t)=i, X^+(t)=j, V^-(t)\in A, V^+(t)\in B, \tau<\infty\} =$$

$$= \frac{h_k \pi_i p_j \int_A e^{\alpha x} Q_{ij}(x+B)\mathrm{d}x}{\pi\left[\int_{\mathbf{R}_+} x e^{\alpha x} Q(\mathrm{d}x)\right] h},$$

provided that the function

$$x \to e^{\alpha x}((Q_{ij})_B 1_A)(x)p_j = e^{\alpha x} Q_{ij}(x+B)1_A(x)p_j$$

is directly Riemann integrable in the sense of Feller. But this is the case since this function is Riemann integrable by assumption and dominated by the directly Riemann integrable function $e^{\alpha x} f_i(x)$. By the same Proposition 1.2,

$$\lim_{t\to\infty} e^{\alpha t} \mathbf{P}_k\{t<\tau<\infty\} = \lim_{t\to\infty} e^{\alpha t} \sum_{l\in I} R_{kl} * f_2(t)$$

$$= \left[\pi\left[\int_{\mathbf{R}_+} x e^{\alpha x} Q(\mathrm{d}x)\right] h\right]^{-1} h_k \left[\pi \int_{\mathbf{R}_+} e^{\alpha x} f(x)\mathrm{d}x\right].$$

The assertion now follows.

We get as immediate corollaries:

Corollary 4.2. *The quasi-stationary limit*

$$\mu_i^{(b)}(A) = \lim_{t\to\infty} \mathbf{P}_k\{X(t)=i, V^-(t)\in A \mid t<\tau<\infty\}$$

exists independently of $k\in I$ provided that $A\in \mathcal{R}_+$ is such that its boundary is a null set, $\ell(\partial A)=0$, and the function $(m,x)\to e^{\alpha x} f_m(x)$ is directly Riemann integrable w.r.t. π. We have

$$\mu_i^{(b)}(A) = \left[\pi \int_{\mathbf{R}_+} e^{\alpha x} f(x)\mathrm{d}x\right]^{-1} \pi_i \int_A e^{\alpha x} f_i(x)\mathrm{d}x. \tag{1}$$

Proof. $\ell(\partial A)=0$ implies that $x\to 1_A(x)$ is continuous a.e. The rest of the proof is analogous to the proof of Theorem 4.1.

Corollary 4.3. *The quasi-stationary limit*

$$\mu_{ij}^{(f)}(B) = \lim_{t\to\infty} \mathbf{P}_k\{X(t)=i, X^+(t)=j, V^+(t)\in B \mid t<\tau<\infty\}$$

exists independently of $k\in I$ provided that the function $x\to Q_{ij}(x+B)$ is Riemann *integrable and the function $(m,x)\to e^{\alpha x}f_m(x)$ is directly Riemann integrable w.r.t. π. We have*

$$\mu_{ij}^{(f)}(B)=\left[\pi\int_{\mathbf{R}_+} e^{\alpha x}f(x)\mathrm{d}x\right]^{-1}\pi_i p_i\int_{\mathbf{R}_+} e^{\alpha x}Q_{ij}(x+B)\mathrm{d}x. \tag{2}$$

Proof. The conditions of Theorem 4.1 are satisfied when $A=\mathbf{R}_+$.

It is easy to verify, by considering the limiting marginal distribution of $\{X(t)\}$ in (1) and (2), that the resulting expressions for $\lim_{t\to\infty}\mathbf{P}_k\{X(t)=i\mid t<\tau<\infty\}$ both coincide with the expression obtained in Theorem 6 of [9].

As we mentioned in the introduction, an irreducible α-positive recurrent Markov process has an (up to scalar multiplication) unique α-invariant vector $\pi=(\pi_i)_{i\in I}$ and a unique I-type quasi-stationary distribution $\mu=(\mu_i)_{i\in I}$, and these are linked together by the equivalent equations $\mu_i=(\sum_{j\in I}\pi_jp_j)^{-1}\pi_ip_i$ and $\pi_i=$ (constant) $\times\, p_i^{-1}\mu_i$. Guided by this and the above corollaries we define $\nu^{(b)}$ as the measure on $I\times\mathbf{R}_+$, which assigns the value

$$\nu_i^{(b)}(A)=\pi_i\int_A e^{\alpha x}(1-\bar{B}_i)(x)\mathrm{d}x \tag{3}$$

to the rectangle $\{i\}\times A$, and $\nu^{(f)}$ as the measure on $I\times\bar{I}\times\mathbf{R}_+$, which assigns the value

$$\nu_{ij}^{(f)}(B)=\pi_i\int_{\mathbf{R}_+} e^{\alpha x}Q_{ij}(x+B)\mathrm{d}x \tag{4}$$

to the rectangle $\{i\}\times\{j\}\times B$. We will then show that $\nu^{(b)}$ is α-invariant for $P_t^{(b)}$ and that $\nu^{(f)}$ is α-invariant for $P_t^{(f)}$, where $P_t^{(b)}$ (resp. $P_t^{(f)}$) is the restriction of the transition probability $\bar{P}_t^{(b)}$ ($\bar{P}_t^{(f)}$) to $I\times\mathbf{R}_+$ ($I\times\bar{I}\times\mathbf{R}_+$).

Note that if $p_i\equiv1$ the right-hand sides in (1) and (3) are the same, up to a scalar multiplication, and so are the right-hand sides in (2) and (4). We shall see below that $\nu^{(b)}$ and $\nu^{(f)}$ can only be finite measures if $p_i\equiv1$.

Theorem 4.4. (i) *For all $t\in\mathbf{R}_+$,*

$$\nu^{(b)}P_t^{(b)}=e^{-\alpha t}\nu^{(b)}.$$

(ii) *$\nu^{(b)}(I\times\mathbf{R}_+)<\infty$ implies $p_i\equiv1$.*

Before proving this theorem we discuss an interesting consequence of it. Suppose that $\nu^{(b)}(I\times\mathbf{R}_+)<\infty$ so that $\mu^{(b)}=[\nu^{(b)}(I\times\mathbf{R}_+)]^{-1}\nu^{(b)}$ and

$$\mu^{(b)}P_t^{(b)}=e^{-\alpha t}\mu^{(b)},\quad t\in\mathbf{R}_+.$$

Denote by $\mathbf{P}_\mu^{(b)}$ that canonical probability measure on the sample space of the backward process $\{(X(t),V^-(t))\}$ which corresponds to the initial probability $\mu^{(b)}$. When considering sets $\{i\}\times\mathbf{R}_+$ we may write the left-hand side $\mu^{(b)}P_t^{(b)}(\{i\}\times\mathbf{R}_+)$ as $\mathbf{P}_\mu^{(b)}\{X(t)=i\}$ and consequently the equation as

$$\mathbf{P}_\mu^{(b)}\{X(t)=i\}=e^{-\alpha t}\mu_i^{(b)}(\mathbf{R}_+),\qquad i\in I,\quad t\in\mathbf{R}_+.$$

This can be taken as a stationarity result "under uniform exponential dying rate". Similarly by considering the set $I \times \mathbf{R}_+$, and since $\{X(t) \in I\} = \{\tau > t\}$, we find

$$\mathbf{P}_\mu^{(h)}\{\tau > t\} = e^{-\alpha t}, \quad t \in \mathbf{R}_+.$$

Using the terminology of Keilson [6] this expresses that the "quasi-stationary exit time is exponentially distributed". Compare also with Theorem 4 of Keilson [7].

Proof. (i) Fix $j \in I$, $B \in \mathcal{R}_+$.

$$e^{\alpha t}(\nu^{(b)}P_t^{(b)})(\{j\} \times B) =$$

$$= e^{\alpha t} \sum_{i \in I} \int_{\mathbf{R}^+} \pi_i e^{\alpha x}(1 - \bar{B}_i(x))\mathrm{d}x (1 - \bar{B}_i(x))^{-1}$$

$$\times \Bigg[\sum_{k \in I} \int_{[0,t)} 1_B(y) \int_{(x,t+x-y]} \mathrm{d}Q_{ik}(u)\mathrm{d}R_{kj}(t + x - y - u)(1 - \bar{B}_j(y))$$

$$+ \delta_{ij} 1_B(t + x)(1 - \bar{B}_i(t + x)) \Bigg]$$

$$= \int_{\mathbf{R}_+} \mathrm{d}x \int_{[0,t)} 1_B(y) \int_{(x,t+x-y]} (\pi \mathrm{d}Q^\alpha(u)\mathrm{d}R^\alpha(t + x - y - u))_j (1 - \bar{B}_j)^\alpha(y)$$

$$+ \pi_j \int_{\mathbf{R}_+} 1_B(t + x)(1 - \bar{B}_j)^\alpha(t + x)\mathrm{d}x$$

$$= \mathrm{I} + \mathrm{II};$$

the proof follows from the observation that

$$\mathrm{I} = \int_{\mathbf{R}_+} \mathrm{d}x \int_{[0,t)} 1_B(y) \int_{(0,t-y]} (\pi \mathrm{d}Q^\alpha(u + x)\mathrm{d}R^\alpha(t - y - u))_j (1 - \bar{B}_j)^\alpha(y)$$

$$= \int_{[0,t)} 1_B(y) \int_{(0,t-y]} \mathrm{d}u(\pi Q^\alpha[u, \infty))\mathrm{d}R^\alpha(t - y - u))_j (1 - \bar{B}_j)^\alpha(y)$$

$$= \int_{[0,t)} 1_B(y)\mathrm{d}\ell(t - y)\pi_j (1 - \bar{B}_j)^\alpha(y) \qquad \text{by Lemma 3.1}$$

$$= \pi_j \int_{[0,t) \cap B} e^{\alpha y}(1 - \bar{B}_j)(y)\mathrm{d}y;$$

$$\mathrm{II} = \pi_j \int_{[t,\infty) \cap B} e^{\alpha x}(1 - \bar{B}_j)(x)\mathrm{d}x.$$

(ii) Assume, to produce a contradiction, that $\nu^{(b)}(I \times \mathbf{R}_+) < \infty$ and $p_k < 1$ for all $k \in I$ (see the remark made before Theorem 4.1). Since Q is non-degenerate there exist $j, k \in I$ such that $Q_{jk}(0, \infty) > 0$. By (i) we get

$$\int_{\mathbf{R}_+} \mathrm{d}\nu_j^{(b)}(x)\mathbf{P}_{(j,x)}^{(b)}\{\tau = \infty\} =$$

$$= \lim_{t \to \infty} \int_{\mathbf{R}_+} \mathrm{d}\nu_j^{(b)}(x)\mathbf{P}_{(j,x)}^{(b)}\{\tau > t\}$$

$$= \lim_{t\to\infty} \int_{\mathbf{R}_+} d\nu_j^{(b)}(x) P_t^{(b)}((j,x), I \times \mathbf{R}_+)$$

$$\leq \lim_{t\to\infty} e^{-\alpha t} \nu^{(b)}(I \times \mathbf{R}_+) = 0.$$

On the other hand,

$$\int_{\mathbf{R}_+} d\nu_j^{(b)}(x) \mathbf{P}_{(j,x)}^{(b)}\{\tau = \infty\} =$$

$$= \int_{\mathbf{R}_+} d\nu_j^{(b)}(x) \mathbf{P}_j\{\tau = \infty, V^+(0) \geq x\} / \mathbf{P}_j\{V^+(0) \geq x\}$$

$$\geq \int_{\mathbf{R}_+} \pi_j e^{\alpha x} dx \int_{(x,\infty)} dQ_{jk}(y)(1 - p_k) > 0,$$

since $Q_{jk}(0,\infty) > 0$, $p_k < 1$.

Theorem 4.5. (i) *For all* $t \in \mathbf{R}_+$

$$\nu^{(f)} P_t^{(f)} = e^{-\alpha t} \nu^{(f)},$$

(ii) $\nu^{(f)}(I \times \bar{I} \times \mathbf{R}_+) < \infty$ implies $p_i \equiv 1$.

Again, with self-evident notation, we have the following consequent results: If $\nu^{(f)}(I \times \bar{I} \times \mathbf{R}_+) < \infty$, then

$$\mathbf{P}_\mu^{(f)}\{X(t) = i\} = e^{-\alpha t} \mu_i^{(f)}(\mathbf{R}_+), \qquad i \in I, \quad t \in \mathbf{R}_+,$$

and

$$\mathbf{P}_\mu^{(f)}\{\tau > t\} = e^{-\alpha t}, \quad t \in \mathbf{R}_+.$$

Proof. (i) Fix $k \in I$, $l \in \bar{I}$, $A \in \mathcal{R}_+$, $B \in \mathcal{R}_+ (B \subset (0,\infty))$.

$$e^{\alpha t}(\nu^{(f)} P_t^{(f)})(\{k\} \times \{l\} \times B) =$$

$$= e^{\alpha t} \sum_{i \in I} \sum_{j \in \bar{I}} \int_{y \in (0,\infty)} \int_{x \in \mathbf{R}_+} \pi_i e^{\alpha x} dQ_{ij}(x+y) dx$$

$$\times [R_{jk} * (Q_{kl})_B (t-y) + \delta_{ik}\delta_{jl} 1_B(y-t)]$$

$$= \sum_{j \in \bar{I}} \int_{y \in (0,\infty)} \int_{x \in \mathbf{R}_+} dx (\pi dQ^\alpha(x+y))_j (R_{jk}^\alpha * (Q_{kl})_B^\alpha)(t-y)$$

$$+ \pi_k \int_{y \in (0,\infty)} \int_{x \in \mathbf{R}_+} e^{\alpha(t+x)} dQ_{kl}(x+y) dx\, 1_B(y-t)$$

$$= \mathrm{I} + \mathrm{II};$$

the proof follows since

$$\mathrm{I} = \sum_{j \in I} \int_{y \in (0,\infty)} dy (\pi Q^\alpha [y,\infty))_j (R_{jk}^\alpha * (Q_{kl})_B^\alpha)(t-y) \qquad \text{since } R_{ik} = 0,$$

$$= \pi_k \int_{[0,t]} (Q_{kl})^{\alpha}_{B}(x)\mathrm{d}x \qquad \text{by Lemma 3.1.}$$

$$= \pi_k \int_{[0,t]} \mathrm{e}^{\alpha x} Q_{kl}(x + B)\mathrm{d}x\,;$$

$$\mathrm{II} = \pi_k \int_{\mathbf{R}_+} \mathrm{e}^{\alpha(t+x)} Q_{kl}(x + t + B)\mathrm{d}x$$

$$= \pi_k \int_{[t,\infty)} \mathrm{e}^{\alpha x} Q_{kl}(x + B)\mathrm{d}x.$$

(ii) Use a similar argument as in the proof of Theorem 4.4(ii).

5. A special case: the terminating renewal process

It is of some interest to study the application of the preceding results to terminating renewal processes.

A terminating renewal process is identical to a two-state Markov renewal process. Let these states be 0 and ∂. The process terminates when it reaches ∂. For the transition probability from 0 we write

$$Q_{00} = F, \qquad Q_{0\partial} = G\,;$$

i.e., F denotes the defective distribution ($c = F(\mathbf{R}_+) < 1$) of the consecutive renewal epochs before the termination of the process and G denotes the distribution, having total mass $G(\mathbf{R}_+) = 1 - c$, of a last renewal epoch leading from 0 to ∂. (Our definitions are somewhat more general than those of Feller [5, Ch. XI.6], since Feller assumed that $G = (1-c)\varepsilon_0$ where ε_0 denotes the probability which assigns unit mass to the origin.) In the following we write

$$1 - (F[0,t] + G[0,t]) = h(t), \quad t \in \mathbf{R}_+.$$

We shall consider the backward process $\{V^-(t)\}$ only. By Corollary 4.2 and Theorem 4.4 we get:

Corollary 5.1. *Suppose that F is non-lattice and satisfies the α-positivity conditions*

$$\int_{\mathbf{R}_+} \mathrm{e}^{\alpha x}\,\mathrm{d}F(x) = 1, \qquad \int_{\mathbf{R}_+} \mathrm{e}^{\alpha x}\,\mathrm{d}F(x) < \infty \quad \textit{for some } \alpha > 0.$$

Then

(i) *the quasi-stationary limit*

$$\mu^{(b)}(A) = \lim_{t\to\infty} \mathbf{P}_0\{V^-(t) \in A \mid \tau > t\}$$

$$= \left[\int_{\mathbf{R}_+} \mathrm{e}^{\alpha x} h(x)\mathrm{d}x\right]^{-1} \int_A \mathrm{e}^{\alpha x} h(x)\mathrm{d}x, \quad A \in \mathcal{R}_+,$$

exists, provided that the boundary of $A \in \mathcal{R}_+$ is a null set, i.e. $\ell(\partial A)=0$, and the function $x \to e^{\alpha x}h(x)$ is directly Riemann integrable in the sense of Feller.

(ii) *for all $t \in \mathbf{R}_+$, $A \in \mathcal{R}_+$*

$$\mathbb{P}_\mu^{(b)}\{V^-(t) \in A, \tau > t\} = e^{-\alpha t}\mu(A),$$

and in particular

$$\mathbb{P}_\mu^{(b)}\{\tau > t\} = e^{-\alpha t}.$$

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